Lecture 5: Quiver varieties:

$\mathbb{F}$ = any field (later, $\mathbb{C}$)

Q = quiver $(\underbrace{Q_0}_{\text{vertices}}, \underbrace{Q_1}_{\text{arrows}})$

$(V_i, \{a\})_{i \in Q_0,\, a \in Q_1}$ rep.

$\mathbb{F}Q :=$ path alg: basis = <u>paths</u> in Q

mult: concatenation.

$\mathrm{Rep}(Q) = \mathrm{Rep}(\mathbb{F}Q) :=$ right mods$(\mathbb{F}Q)$.

__Ex__ $\bullet \circlearrowleft$ = "Jordan quiver" "type $\tilde{A}_0$"

$$\mathrm{Rep} = (V, \rho) \quad \rho : V \to V.$$

$$\text{Isoclasses} = \left\{ (n, \mathcal{C}) \;\middle|\; \begin{array}{l} n \geq 0 \\ \mathcal{C} \subseteq \mathrm{Mat}_n \mathbb{F} \\ \text{conj class.} \end{array} \right\}$$

__Ex__ $Q = \bullet \rightrightarrows \bullet$ "Kronecker" & $\left[\begin{array}{l} \text{"type } \tilde{A}_1\text{"} \\ A_1 \;\cdot \end{array}\right]$

Reps: (V, W, ρ_1, ρ_2), $\rho_1, \rho_2 : V \to W$.

Special case: ρ_1 is an isom.

$$\underset{\rho_1 \text{ invertible}}{\text{Reps}/\cong} = (V, \rho_2 \circ \rho_1^{-1})/\cong \, = \text{reps}(\circlearrowleft)/\cong$$

Same: ρ_2 invertible.

<u>Ex:</u> A_2 quiver $\overset{m}{\bullet} \longrightarrow \overset{n}{\bullet}$ Dynkin.

$$\text{Reps}/\cong \;\longleftarrow \{(m,n,r) \mid m,n \geq 0, 0 \leq r \leq \min(m,n)\}$$

Indecomposable reps:

"Krull-Schmidt": every decomposition into indecomposables is isomorphic (finite length)
finite-dimensional

Exer: Not true for vector bundles.
Not finite length

A_2 quiver: 3 indecomposables:

$\mathbb{C} \xrightarrow{0} 0$, $0 \xrightarrow{0} \mathbb{C}$, $\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C}$

Finitely many.

$\mathfrak{sl}_3 \rightsquigarrow 3$ pos roots.

Cases $\circlearrowright$, $\rightrightarrows$: ∞ many indec reps: $|\mathbb{F}| = \infty$.

already of dims 1 , $(1,1)$.

<u>Defn</u> $\dim(V_i, p_a) := (\dim V_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.

<u>Thm (Gabriel)</u>: There are finitely many isoclasses of indecomposables $\Longleftrightarrow$ Q is ADE Dynkin.

In this case, the dims of indecs $\longleftrightarrow$ pos roots of Q.

$\exists!$ indec of each such dim.

Rmk: If we fix a dim vector then if $|\mathbb{F}| = \infty$, we still in general get ∞ many iso classes if Q not Dynkin.

Thm (Kac) Q = any quiver without loops

dims (indecs) = pos roots of KM alg $\mathfrak{g}_Q$.

infinite dim.

- α is pos <u>real</u> root: $\exists!$ indec $/\cong$
- <u>α is pos <u>imag</u> root</u>: ∞ many (if $|\mathbb{F}| = \infty$)

Ex $\bullet \rightrightarrows \bullet$. Real roots: $\underset{m+1}{\bullet} \rightrightarrows \underset{m}{\bullet}$ or $\underset{m}{\bullet} \rightrightarrows \underset{m+1}{\bullet}$

(Pos: coeffs ≥ 0) $\quad m \in \mathbb{Z}$.

Imag roots: $\underset{m}{\bullet} \rightrightarrows \underset{m}{\bullet}$.

<u>Saw:</u> $m=0$ real case: $\exists!$ indec, $m=1$ in imag case: ∞ many.

Exer: extend to general n.

KM alg: $Q \rightsquigarrow A \in \mathrm{Mat}_{Q_0}(\mathbb{N})$

over $\mathbb{C}$

$A + A^t$ is symmetric

$= \text{adj matrix}(\overline{Q})$

$C := 2I - (A + A^t)$

"(Cartan matrix)"

$\mathfrak{h} := \mathbb{C}^{Q_0} \rightsquigarrow (e_i, e_j) = C_{ij},\ i,j \in Q_0.$

Defn $\mathfrak{h} := \mathrm{Span}(H_i)$

$\alpha \in \mathfrak{h}^*$ **root** if $\exists$ weight vector X

$\alpha \neq 0$.

$$[H_i, X] = \alpha(H_i) X.$$

Fact: $\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{wt } 0 \text{ space}} \oplus \bigoplus_{\alpha \text{ root}} \mathfrak{g}_\alpha$ $:=$ span of X as above.

$\underline{\text{Ex}}$ $\Longrightarrow$ type $\hat{A}_1 \rightsquigarrow$ pos roots of $\widehat{\mathfrak{sl}}_2$ are the dims of indecs (by Kac theorem)

root space $(m+1, m)$ spanned by $e \cdot t^m$

$(m, m+1)$ " " $f \cdot t^m$.

$m \neq 0$ (m, m): " " $h \cdot t^m$

$$\boxed{\underline{\mathbb{C}} \cdot c \hookrightarrow \widehat{\mathfrak{sl}}_2 \twoheadrightarrow \mathfrak{sl}_2[t, t^{-1}]}$$

$$\mathfrak{h} = \operatorname{Span}(h, c)$$

Combinatorial description of roots

- Real roots = result of applying <u>reflections</u> to $e_i \in \mathbb{N}^{Q_0}$, $i \in Q_0$

$$S_i \alpha := \alpha - \underset{\text{Cart.}}{(\alpha, e_i)} e_i$$

$$(S_i \alpha)_j = \begin{cases} \alpha_j, & j \neq i \\ -\alpha_i + \sum\limits_{\substack{j \to i \\ \text{or } i \to j}} \alpha_j, & \end{cases} \text{case no loops.}$$

Note: For D_4, there are 12 pos roots, all real (Dynkin $\Longleftrightarrow$ only real)

$\Rightarrow$ 12 <u>parameters</u> in classifying reps of D_4 ($\cong$

$\Rightarrow$ solve "triple of subspace problem"

$$\begin{tikzcd} V_1 \arrow[dr] & & \\ & V & V_3 \arrow[l] \\ V_2 \arrow[ur] & & \end{tikzcd}$$

triples $V_1, V_2, V_3 \subseteq V$.

9 indecs where $V_i \hookrightarrow V$. $\Rightarrow$ 9 parameters.

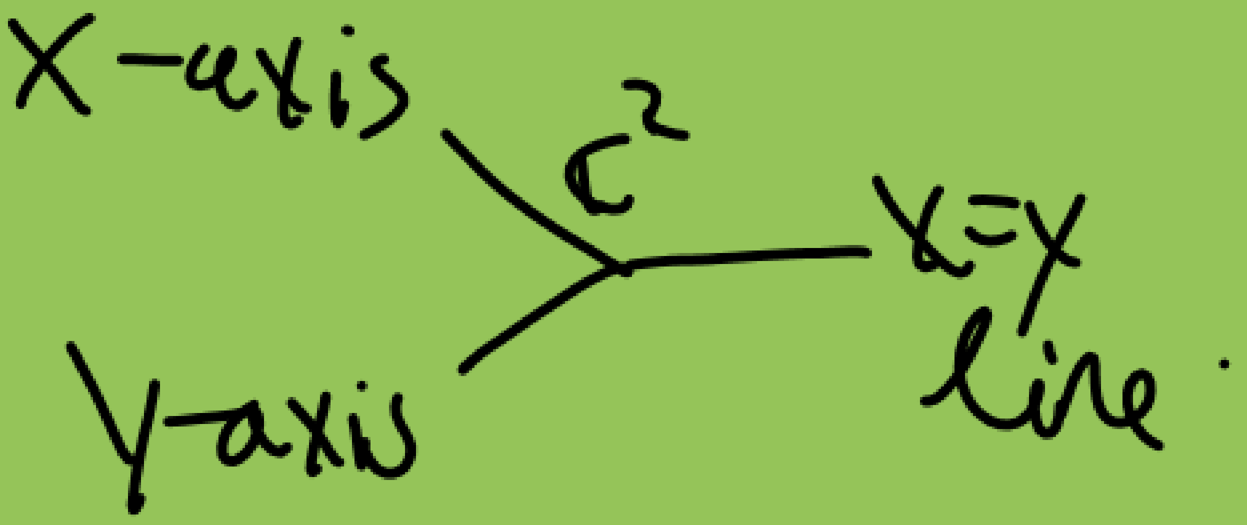
What are the params?

$\dim V$, $\dim V_i$, $\dim V_i \cap V_j$, $\dim V_1 \cap V_2 \cap V_3$,

$\underline{1}$ $\quad 3 \quad 3 \quad$ $\underline{1}$

$\dim V_1 + V_2 + V_3$.

use to rep



<u>Imag roots</u>: vectors in $\mathbb{Z}^{Q_0}$ obtained by reflecting vector in <u>fund region</u>.

<u>Fund region</u>: $\mathcal{F} := \{ v \in \mathbb{N}^{Q_0} \mid (v, e_i)_{\text{Cartan}} \leq 0 \ \forall i, \ \text{supp}(v) \text{ conn.} \}$

$\text{supp}(v) = \{ i \in Q_0 \mid v_i \neq 0 \}$.

<u>Ex</u>  $\tilde{A}_0$: ∞ many isoclasses dim $= 1 = \delta$.

Kronecker: $(1,1) = \delta$: $\exists \infty$ many isoclasses.

Ext Dynkin case (chopping off one vertex is Dynkin)

C is positive semidefinite $(e,e) \geq 0$ $\forall e$.

$\operatorname{rk} C = |Q_0| - 1$

$\Rightarrow \mathcal{I} = \mathbb{Z} \cdot \delta$, δ spans $\ker(C)$.

$(\delta, e_i) = 0$ $\forall i \Rightarrow$ Imag roots $= \mathbb{Z}_{\neq 0} \cdot \delta$.

Jordan: $\circlearrowleft_m$ all imag. | $m \rightrightarrows m$ are imag roots.

Given $n \geq 1$, indec conj classes in $\mathrm{Mat}_n \mathbb{F}$, $\mathbb{F} = \overline{\mathbb{F}}$, are:

$$\left[\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}\right] \Rightarrow \infty \text{ many if } |\mathbb{F}| = \infty.$$

As before, $\mathbb{F}^m \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} \mathbb{F}^m$.

ρ_1 isom $\Rightarrow$ get reps of Jordan

$\Rightarrow \infty$ many isoclasses of indecs if $|\mathbb{F}| = \infty$.

Ex:

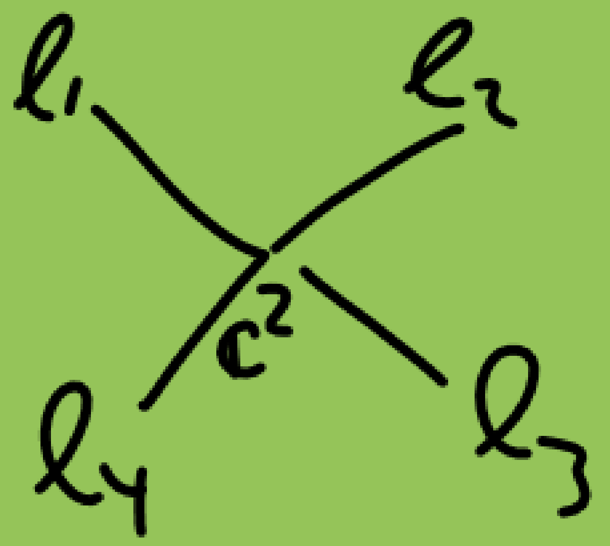
$$\begin{tikzcd} & & 1 \arrow[dll] & & \\ 1 \arrow[r] & 1 \arrow[r] & 1 \arrow[r, dash] & \cdots \arrow[r] & 1 \arrow[ull] \end{tikzcd}$$

$\widetilde{A}_n$

Going around: Comp $= \lambda \cdot \mathrm{Id}$,

$\lambda =$ continuous invt.

$\tilde{D}_4$: quadruples of subspaces problem:



$\rightsquigarrow$ cont. invt: <u>cross-ratio</u> of $\ell_i \in \mathbb{CP}^1$.

GL_2 takes any triple of lines in $\mathbb{C}^2$ to

x-axis, y-axis, $x=y$ line

(on $\mathbb{CP}^1$, PGL_2 take 3 pts to $0, 1, \infty$)

Quiver varieties:

Moduli of quiver reps:

$$\mathrm{Rep}_\alpha Q := \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{a_t}}, \mathbb{C}^{\alpha_{a_h}})$$

$a : a_t \longrightarrow a_h$

$\shortparallel$ all reps with v.s. $\mathbb{C}^{\alpha_i}$ $\forall i \in Q_0$

$$M_\alpha Q := \mathrm{Rep}_\alpha Q /\!/ GL_\alpha := \prod GL_{\alpha_i}.$$

$M_\alpha Q$ is finite $\forall \alpha$ if Q Dynkin
(0-dim)

infinite (pos-dim) for Q non-Dynkin

if $\alpha \geq \beta =$ some imag root.

<u>Symplectic version:</u> " $T^* M_\alpha Q$ " if stacks.

$T^* \mathrm{Rep}_\alpha Q /\!/\!/_{\lambda,\theta} GL_\alpha$ Ham red. $\lambda \in \mathbb{C}^{Q_0}$ def.
$\theta \in \mathbb{Z}^{Q_0}$.